Growth processes related to the dispersionless Lax equations *

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Abstract

This paper is a short review of the connection between certain types of growth processes and the integrable systems theory, written from the viewpoint of the latter. Starting from the dispersionless Lax equations for the 2D Toda hierarchy, we interpret them as evolution equations for conformal maps in the plane. This provides a unified approach to evolution of smooth domains (such as Laplacian growth) and growth of slits. We show that the Löwner differential equation for a parametric family of conformal maps of slit domains arises as a consistency condition for reductions of the dispersionless Toda hierarchy. It is also demonstrated how the both types of growth processes can be simulated by the large N limit of the Dyson gas picture for the model of normal random matrices.

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1 Introduction

Growth problems of Laplacian type (such as Hele-Shaw viscous flows) refer to dynamics of a moving front (an interface) between two distinct phases driven by a harmonic scalar field. These essentially nonlinear and non-local problems attract much attention for quite a long time [1, 2].

Remarkably, Laplacian growth with vanishing surface tension possesses an integrable structure uncovered in [3]. Since evolution of planar domains is most naturally described by time-dependent conformal maps, there is no surprise that this structure is actually immanent for general conformal maps and classical boundary value problems. Specifically, in [4] it has been shown that evolution of conformal maps is governed by an integrable hierarchy of nonlinear partial differential equations which is a zero dispersion version [5] of the 2D Toda hierarchy [6]. In fact the Lax equations for this hierarchy can be derived from the classical theory of conformal maps depending on parameters.

In the present paper we have tried to give a short review of these and related developments in the context of the integrable systems theory. Taking the Lax representation of the dispersionless 2D Toda (dToda) hierarchy as a starting point, we follow how it induces the contour dynamics and dynamics of conformal maps. The Lax function, which is supposed to be univalent in some neighborhood of infinity, is interpreted as a conformal map from a fixed reference domain to the complement of the growing domain.

Depending on the type of solutions to the dToda hierarchy, the growing domains can be either smooth or singular like cuts or slits. Hence the system of dispersionless Lax equations serves as a master dynamical equation not only for Laplacian growth but also for growth of slits. Conformal maps of parametric families of slit domains are known to satisfy a differential equation proposed by K.Löwner in 1923 [7]. Its connection with nonlinear integrable equations was pointed out in [8], see also [9, 10, 11]. Following [12], we demonstrate how the radial Löwner equation arises in the context of the dToda hierarchy.

The unified treatment of the two types of growth processes in the framework of the Toda integrable system, which we emphasize in this paper, seems to be especially promising in the light of the stochastic Löwner evolution (SLE) approach [13]. This might give a hint how to incorporate, in an intelligent way, a stochastic ingredient into growth problems of Laplacian type.

An instructive representation of solutions to the dToda hierarchy is provided by the large N limit of certain matrix integrals or their eigenvalue versions (the Dyson gas representation), with the associated growth processes being simulated by evolution of support of eigenvalues. In the last section we outline the Dyson gas representation for Laplacian growth and growth of slits.

Section 2 contains the necessary material on the dToda hierarchy and its Lax representation. In section 3 we present the general solution to the hierarchy in terms of canonical transformations and distinguish the classes of non-degenerate and degenerate solutions. In section 4 we associate a contour dynamics with any solution to the Lax equations. A particular subclass of degenerate solutions is studied in section 5, where the Löwner equation is derived from the Lax equations. Finally, section 6 contains the Dyson gas representation for both non-degenerate and degenerate solutions.

2 Lax representation for the dToda hierarchy

This section contains a standard material which we present in a form convenient for our purposes. For a more complete account of dispersionless hierarchies, their algebraic structure, solutions and applications see [5],[14]-[17].

Dispersionless Lax equations. We start with the Lax representation of the dToda hierarchy with certain reality conditions imposed. The main object is the Lax function $z(w)$ represented as a Laurent series of the form

$$z(w) = rw + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \dots$$

The leading coefficient r is assumed to be real while all other coefficients a_i are in general complex numbers. All the coefficients depend on deformation parameters (or "times") t_0 (a real number) and $t_1, t_2, t_3, \dots$ (complex numbers) in accordance with the Lax equations

$$\frac{\partial z(w)}{\partial t_k} = \{A_k(w),\, z(w)\}\,, \quad \frac{\partial z(w)}{\partial \bar{t}_k} = -\{\bar{A}_k(w^{-1}),\, z(w)\} \tag{1}$$

where for any two functions of w, t_0 we set

$$\{f,\, g\} := \frac{\partial f}{\partial \log w}\frac{\partial g}{\partial t_0} - \frac{\partial f}{\partial t_0}\frac{\partial g}{\partial \log w} \tag{2}$$

Here and below the bar is complex conjugation and $\bar{f}(w)$ means $\overline{f(\bar{w})}$. The reality condition thus implies that the second half of the Lax equations (with $\bar{t}_k$-derivatives) is obtained from the first one by complex conjugation with w on the unit circle. The generators of the flows are constructed as follows:

$$A_k(w) = \Big(z^k(w)\Big)_+\,, \quad A_0(w) = \log w$$

For the dToda hierarchy, the $(\dots)_+$-operation is

$$\Big(z^k(w)\Big)_+ := \Big(z^k(w)\Big)_{>0} + \frac{1}{2}\Big(z^k(w)\Big)_0$$

Hereafter, $(\dots)_S$ means taking the terms of the Laurent series with degrees belonging to the subset $S \in \mathbb{Z}$ (in particular, $(\dots)_0$ is the free term). Note that at $k = 0$ equations (1) become tautological identities. The second Lax function of the dToda hierarchy is $\bar{z}(w^{-1})$. The reality conditions (i.e. the requirement that its coefficients are complex conjugate to those of the $z(w)$) imply that it obeys the same Lax equations. It should be noted that (1) is just a compact form of writing evolution equations with respect to *real* deformation parameters $t_k^{\mathrm{R}} = \mathcal{R}e\, t_k$, $t_k^{\mathrm{I}} = \mathcal{I}m\, t_k$:

$$\frac{\partial z(w)}{\partial t_k^{\mathrm{R}}} = \Big\{A_k(w) - \bar{A}_k(w^{-1}),\, z(w)\Big\}\,, \quad \frac{\partial z(w)}{\partial t_k^{\mathrm{I}}} = i\Big\{A_k(w) + \bar{A}_k(w^{-1}),\, z(w)\Big\}$$

We are especially interested in the class of solutions such that $z(w)$, for all t_k in an open set of the space of parameters, is a univalent function in a neighborhood of

infinity including the exterior of the unit circle. This means that in this neighborhood $z(w_1) = z(w_2)$ if and only if $w_1 = w_2$. From now on, we assume that $z(w)$ belongs to this class. In this case $z(w)$ is a conformal map from the exterior of the unit circle to a domain in the complex plane containing infinity while $\bar{z}(w^{-1})$ is a conformal map from the interior of the unit circle to the complex conjugate domain.

Let $w(z)$ be the inverse function to the Lax function $z(w)$. In terms of the inverse function, the evolution equations (1) acquire a simpler form:

$$\frac{\partial \log w(z)}{\partial t_k} = \frac{\partial A_k}{\partial t_0}, \quad \frac{\partial \log w(z)}{\partial \bar{t}_k} = -\frac{\partial \bar{A}_k}{\partial t_0} \tag{3}$$

Here $A_k = A_k(w(z))$, $\bar{A}_k = \bar{A}_k(1/w(z))$ are regarded as functions of z, and the derivatives are taken at constant z.

By purely algebraic manipulations, one can show [5] that the compatibility conditions for the Lax equations (1) read

$$\partial_{t_j} A_k(w) - \partial_{t_k} A_j(w) + \{A_k(w), A_j(w)\} = 0$$

$$\partial_{t_j} \bar{A}_k(w^{-1}) + \partial_{\bar{t}_k} A_j(w) + \{\bar{A}_k(w^{-1}), A_j(w)\} = 0$$

which is a dispersionless version of the "zero curvature" representation. Treating A_k's as functions of z, one can rewrite them in the form similar to (3):

$$\frac{\partial A_j}{\partial t_k} = \frac{\partial A_k}{\partial t_j}, \quad \frac{\partial A_j}{\partial \bar{t}_k} = -\frac{\partial \bar{A}_k}{\partial t_j} \tag{4}$$

Note that at $j = 0$ this system coincides with (3).

Generating form of the Lax equations. Using a generating function of the polynomials A_k, the infinite hierarchy (1) can be represented as a couple of "generating equations". The generating function is defined as

$$\sum_{k\geq 1} \frac{z_1^{-k}}{k} A_k(w) = \sum_{k\geq 1} \frac{\left(z^k(w)\right)_+}{k z^k(w_1)} = -\left[\log\left(1 - \frac{z(w)}{z(w_1)}\right)\right]_+$$

where $w_1 = w(z_1)$. To separate the polynomial part and the free term, we write

$$\log\left(1 - \frac{z(w)}{z(w_1)}\right) = \log\left(1 - \frac{w}{w(z_1)}\right) + \log\frac{r w(z_1)}{z_1} + \log\frac{z(w_1) - z(w)}{r(w_1 - w)}$$

and notice that the expansion of the first (third) term contains only positive (respectively, negative) powers of w while the rest is just the free term. Therefore,

$$\sum_{k\geq 1} \frac{z_1^{-k}}{k} A_k(w) = -\log(w(z_1) - w) + \frac{1}{2}\log\frac{z_1 w(z_1)}{r} \tag{5}$$

Similarly,

$$\sum_{k\geq 1} \frac{\bar{z}_1^{-k}}{k} \bar{A}_k(w^{-1}) = -\log(\bar{w}(\bar{z}_1) - w^{-1}) + \frac{1}{2}\log\frac{\bar{z}_1 \bar{w}(\bar{z}_1)}{r} \tag{6}$$

and the generating Lax equations read

$$D(z_1)z(w) = -\left\{\log(w(z_1)-w) + \frac{1}{2}\log\frac{r}{w(z_1)}\,,\, z(w)\right\} \tag{7}$$

$$\bar{D}(\bar{z}_1)z(w) = \left\{\log(\bar{w}(\bar{z}_1)-w^{-1}) + \frac{1}{2}\log\frac{r}{\bar{w}(\bar{z}_1)}\,,\, z(w)\right\} \tag{8}$$

where we have introduced the differential operators

$$D(z) = \sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_{t_k}\,, \quad \bar{D}(z) = \sum_{k\geq 1}\frac{z^{-k}}{k}\,\partial_{\bar{t}_k}$$

Expanding both sides in powers of z_1, one recovers eqs. (1). In terms of the inverse function, the generating Lax equations acquire a more transparent form (cf. (3)):

$$D(z_1)\log w(z_2) = -\partial_{t_0}\log\left(w(z_1)-w(z_2)\right) + \frac{1}{2}\partial_{t_0}\log\frac{w(z_1)}{r} \tag{9}$$

$$\bar{D}(\bar{z}_1)\log w(z_2) = \partial_{t_0}\log\left(\bar{w}(\bar{z}_1)-w^{-1}(z_2)\right) - \frac{1}{2}\partial_{t_0}\log\frac{\bar{w}(\bar{z}_1)}{r} \tag{10}$$

Tending $z_2 \to \infty$, we obtain the useful equations

$$D(z)\log r = -\frac{1}{2}\partial_{t_0}\log\left(rw(z)\right)\,, \tag{11}$$

$$\bar{D}(\bar{z})\log r = -\frac{1}{2}\partial_{t_0}\log\left(r\bar{w}(\bar{z})\right) \tag{12}$$

Plugging them back into eqs. (9), (10), one can represent the latter relations in a slightly more compact form:

$$D(z_1)\log(rw(z_2)) = -\partial_{t_0}\log\left(rw(z_1)-rw(z_2)\right) \tag{13}$$

$$\bar{D}(\bar{z}_1)\log(rw(z_2)) = \partial_{t_0}\log\left(1-\frac{1}{\bar{w}(\bar{z}_1)w(z_2)}\right) \tag{14}$$

The construction of A_k's implies that the expansion of $A_k(w(z))$ in a Laurent series in z looks like $A_k = z^k + O(1)$. Furthermore, the compatibility conditions (4) and relation (5) allow one to represent the coefficients in the form

$$A_0(w(z)) = \log w(z) = -\frac{1}{2}\,\partial_{t_0}v_0 - \sum_{k\geq 1}\frac{\partial_{t_0}v_k}{k}\,z^{-k} \tag{15}$$

$$A_j(w(z)) = z^j - \frac{1}{2}\,\partial_{t_j}v_0 - \sum_{k\geq 1}\frac{\partial_{t_j}v_k}{k}\,z^{-k}\,, \quad j\geq 1 \tag{16}$$

$$\bar{A}_j(w^{-1}(z)) = \frac{1}{2}\,\partial_{\bar{t}_j}v_0 + \sum_{k\geq 1}\frac{\partial_{\bar{t}_j}v_k}{k}\,z^{-k}\,, \quad j\geq 1 \tag{17}$$

where v_k are functions of the times such that $\partial_{t_j}v_k = \partial_{t_k}v_j$, $\partial_{t_j}\bar{v}_k = \partial_{\bar{t}_k}v_j$. The latter conditions allow one to introduce a real-valued function $\mathcal{F}$ via $v_k = \partial_{t_k}\mathcal{F}$. Equations (13), (14) then become the dispersionless Hirota equations for the function $\mathcal{F}$.

3 General solution to the Lax equations

A general solution to the differential equations (1) is available in an implicit form [5]. To present it, we need an extended version of the Lax formalism.

The idea is as follows. By the definition of the Poisson bracket, $\log w$ and t_0 form a canonical pair: $\{\log w,\, t_0\} = 1$. The evolution according to the Lax equations can be regarded as a t_k-dependent canonical transformation from the pair $(\log w, t_0)$ to another canonical pair whose first member is $\log z(w)$. It is quite natural to introduce the second member which we denote by M. Depending on the situation, we shall treat it either as a function of z and t_0 or as a function of w and t_0 through the composition $M = M(z(w,t_0),t_0)$ (it also depends on the deformation parameters t_k). To find what is M, we note that the condition $\{\log z,\, M\} = 1$ can be identically rewritten as $\partial_{t_0}M(z) = z\partial_z \log w(z,t_0)$. This determines M up to a term depending only on z. The latter is fixed if one requires M to obey the same Lax equations (1). To wit, equation $\partial_{t_k}M = \{A_k, M\}$ (where the derivatives are taken at constant w) is equivalent to

$$\partial_{t_k}M(z) = w\partial_w A_k\, \partial_{t_0}M(z) = z\partial_z A_k$$

Taking into account (15), (16), we can write

$$M = \sum_{k\geq 1} kt_k z^k(w) + t_0 + \sum_{k\geq 1} v_k z^{-k}(w) \tag{18}$$

It is the quasiclassical (dispersionless) limit of the Orlov-Shulman operator [18]. Its geometric meaning depends on the choice of a particular solution. In a similar way, one can construct the conjugate Orlov-Shulman function, $\bar{M}(\bar{z})$, such that the transformation $(\log w, t_0) \to (\log \bar{z}^{-1}(w^{-1}), \bar{M}(\bar{z}(w^{-1}))$ is canonical and $\bar{M}$ obeys the same Lax equations. The Lax equations imply that the composition of the canonical transformations

$$(\log z, M) \to (\log w, t_0) \to (\log \bar{z}^{-1}, \bar{M})$$

does not depend on t_k, i.e., it is an integral of motion. Moreover, any t_k-independent canonical transformation $(\log z, M) \to (\log \bar{z}^{-1}, \bar{M})$ between the Laurent series of the form prescribed above generates a solution to the dToda hierarchy. A detailed proof can be found in [5].

More precisely, let $(\log f(w,t_0), g(w,t_0))$ be a canonical pair: $\{\log f, g\} = 1$. Suppose that the functions $z, \bar{z}, M, \bar{M}$ of the form as above are connected by the functional relations

$$1/\bar{z}(w^{-1}) = f\left(z(w), M(z(w))\right)\,, \quad \bar{M}(\bar{z}(w^{-1})) = g\left(z(w), M(z(w))\right) \tag{19}$$

Then the function $z(w)$ obeys the hierarchy of the Lax equations and its coefficients (as functions of t_k's) thus obey the dToda hierarchy. Conversely, any solution of the dToda hierarchy admits a representation of this form with some (f,g)-pair. Note that the reality conditions imply the following constraints on the functions f, g:

$$\bar{f}^{-1}\left(f^{-1}(w,t_0),\, g(w,t_0)\right) = w\,, \quad \bar{g}\left(f^{-1}(w,t_0),\, g(w,t_0)\right) = t_0 \tag{20}$$

which is a sort of the "unitarity condition" for the canonical transformation $(\log w, t_0) \to (\log f(w,t_0), g(w,t_0))$.

This construction can be made more explicit by introducing the generating function of the canonical transformation $(\log w, t_0) \to (\log f, g)$. An important class of solutions corresponds to the canonical transformations $(\log z, M) \to (\log \bar{z}^{-1}, \bar{M})$ defined by means of a generating function $U(z, \bar{z})$ [19]:

$$M = z\partial_z U(z, \bar{z})\,, \quad \bar{M} = \bar{z}\partial_{\bar{z}} U(z, \bar{z}) \tag{21}$$

Here $U(z, \bar{z})$ can be an arbitrary differentiable real-valued function of z, $\bar{z}$. This form of the canonical transformation implies that the functions $z(w)$ and $\bar{z}(w^{-1})$ are algebraically independent. (By algebraic dependence we mean here existence of a t_k-independent function of two variables $R(z, \bar{z})$ such that $R(z(w), \bar{z}(w^{-1})) = 0$ for all t_k.) These are solutions of generic type. We call them *non-degenerate*. For non-degenerate solutions the "string equation"

$$\{z(w),\, \bar{z}(w^{-1})\} = \frac{1}{U_{z\bar{z}}(z(w), \bar{z}(w^{-1}))} \tag{22}$$

where $U_{z\bar{z}}(z, \bar{z}) \equiv \partial_z \partial_{\bar{z}} U(z, \bar{z})$ holds true. It is obtained by plugging M from (21) into the canonical relation $\{z, M\} = z$.

The origin of the string equation can be understood in a simpler way as follows. From the Lax equations (1) and the Jacobi identity for the Poisson bracket it follows that $\{z(w), \bar{z}(w^{-1})\}$ obeys the same Lax equations:

$$\partial_{t_k}\{z(w), \bar{z}(w^{-1})\} = \Big\{A_k(w),\, \{z(w), \bar{z}(w^{-1})\}\Big\}$$

Therefore, any relation of the form

$$\{z(w), \bar{z}(w^{-1})\} = \omega(z(w), \bar{z}(w^{-1})) \tag{23}$$

where $\omega(z, \bar{z})$ is an arbitrary t_k-independent function of two variables such that $\bar{\omega}(z, \bar{z}) = \omega(\bar{z}, z)$ is consistent with the hierarchy. The approach based on the canonical transformations clarifies the meaning of this function and makes it clear that in fact any solution obeys a string equation of the form (23).

Canonical transformations such that the functions $z(w)$ and $\bar{z}(w^{-1})$ appear to be algebraically dependent can not be represented in the form (21). They correspond to solutions which we call *degenerate*. For degenerate solutions, the Poisson bracket $\{z, \bar{z}\}$ vanishes. Conversely, the relation $\{z, \bar{z}\} = 0$ implies the algebraic dependence. Indeed, in terms of the function $\bar{z}(w^{-1}(z))$ the Lax equation for $\bar{z}(w^{-1})$ reads:

$$\partial_{t_k} \bar{z}(w^{-1}(z)) = \partial_z A_k \{z, \bar{z}\}$$

(the derivatives are taken at constant z), so at $\{z, \bar{z}\} = 0$ we have $\partial_{t_k} \bar{z}(w^{-1}(z)) = 0$ for all t_k. This just means that the relation between $z(w)$ and $\bar{z}(w^{-1})$ is t_k-independent. A particular subclass of degenerate solutions, together with their geometric interpretation, is discussed below in section 5.

4 Contour dynamics

The Lax equations (1) can be understood as equations of a contour dynamics. The contour is the image of the unit circle, i.e., $z(e^{i\theta})$, $0 \le \theta \le 2\pi$. Let us call it the Lax

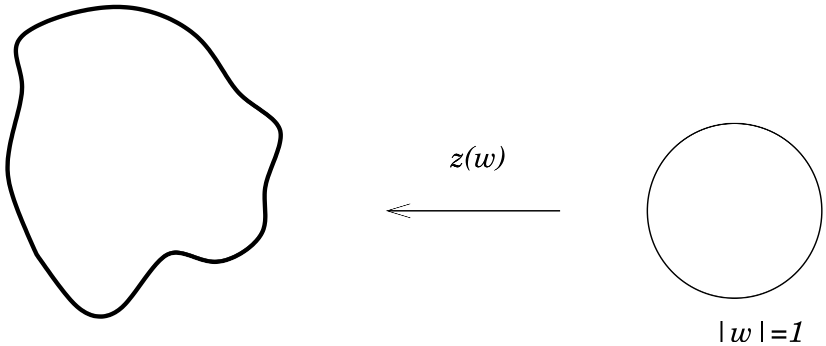

Figure 1: *The Lax contour.*

contour (Fig. 1) and denote it by γ. It depends on the deformation parameters according to the Lax equations.

We need a general kinematic relation. Let $(x(\sigma, t), y(\sigma, t))$ be any parameterizations of a moving contour in the plane, then the normal velocity of the contour points is

$$V_n = \frac{d\sigma}{dl} \left(\partial_\sigma x \partial_t y - \partial_\sigma y \partial_t x \right)$$

where $dl = \sqrt{(dx)^2 + (dy)^2}$ is the line element along the contour.

Applying this formula to the Lax contour $z(e^{i\theta})$ with the specific parametrization $\sigma = \theta$ and $t = t_0$ with all other t_k's fixed, we get the normal velocity of the Lax contour γ at the points $z(w)$, $|w| = 1$:

$$V_n = -\frac{\{z(w),\, \bar{z}(w^{-1})\}}{2|z'(w)|} \tag{24}$$

Here $z'(w) = \partial_w z(w)$ and the Poisson bracket in the numerator is given by (2). More generally, the normal velocities corresponding to the changes of the (real) times $t_k^{\mathrm{R}} = \mathcal{R}e\, t_k$, $t_k^{\mathrm{I}} = \mathcal{I}m\, t_k$ are given by

$$V_n^{(t_k^{\mathrm{R}})} = -\frac{\{z(w),\, \bar{z}(w^{-1})\}}{2|z'(w)|} \left(\phi_k(w) + \bar{\phi}_k(w^{-1}) \right) \tag{25}$$

$$V_n^{(t_k^{\mathrm{I}})} = -\, i\, \frac{\{z(w),\, \bar{z}(w^{-1})\}}{2|z'(w)|} \left(\phi_k(w) - \bar{\phi}_k(w^{-1}) \right) \tag{26}$$

where

$$\phi_k(w) := w \partial_w A_k(w) \tag{27}$$

The function $z(w)$ provides a time-dependent conformal map from the exterior of the unit circle onto the exterior of the Lax contour.

Non-degenerate solutions. The non-degenerate solutions corresponding to the canonical transformation with the generating function $U(z, \bar{z})$ (21) have a clear interpretation in terms of contour dynamics. Eq. (24) together with the string equation (22) states that the normal velocity of the Lax contour at the point $z \in \gamma$ is equal to

$$V_n(z) = -\frac{|w'(z)|}{2\partial_z \partial_{\bar{z}} U(z, \bar{z})}\,, \quad z \in \gamma \tag{28}$$

Eqs. (18), (21) allow us to express the deformation parameters in terms of the moving contour:

$$t_k = \frac{1}{2\pi i k}\oint_{|w|=1} z^{-k-1}(w)M(z(w))dz(w) = \frac{1}{2\pi i k}\oint_{\gamma} z^{-k}\partial_z U\, dz\,, \quad k \geq 1 \tag{29}$$

$$t_0 = \frac{1}{2\pi i}\oint_{|w|=1} M(z(w))d\log z(w) = \frac{1}{2\pi i k}\oint_{\gamma} \partial_z U\, dz \tag{30}$$

We stress that $t_1, t_2, \ldots$ are kept constant, so they are integrals of motion for the contour dynamics (28).

A particularly important case is $U(z,\bar{z}) = z\bar{z}$ which corresponds to the canonical transformation $\bar{z} = z^{-1}M$, $\bar{M} = M$ (i.e., $M = \bar{M} = z\bar{z}$). In this case the normal velocity is given by

$$V_n(z) = -\frac{1}{2}|w'(z)|\,, \quad z \in \gamma \tag{31}$$

Note that $|w'(z)|$ is equal to the normal derivative $\partial_n \log|w(z)|$ of the solution to the Laplace equation with a source at infinity and the Dirichlet boundary condition on the contour. We thus see that (31) is identical to the Darcy law for the dynamics of interface between viscous and non-viscous fluids confined in the radial Hele-Shaw cell, assuming vanishing surface tension on the interface. Formulas (29) state that $t_k = \frac{1}{2\pi i k}\oint_\gamma z^{-k}\bar{z}dz$ are harmonic moments of the exterior of the contour γ. Their conservation in the course of the Laplacian growth dynamics was first established by S.Richardson [20]. Eq. (29) states that the time variable t_0 should be identified with area (divided by π) of the interior domain encircled by γ.

The Laplacian growth with the source at a finite point z_0 corresponds to the same function $U(z,\bar{z}) = z\bar{z}$ and the vector field $\partial_{t_0} + D(z_0) + \bar{D}(\bar{z}_0)$ in the space of deformation parameters. Indeed, using eqs. (25), (26) and relations (5), (6) we obtain the normal velocity

$$V_n(z) = \frac{\partial_n G(z,z_0)}{2\partial_z\partial_{\bar{z}}U(z,\bar{z})}\,, \quad z \in \gamma \tag{32}$$

where

$$G(z,z_0) = \log\left|\frac{w(z)-w(z_0)}{1-w(z)\overline{w(z_0)}}\right|$$

is the Green function of the Dirichlet boundary value problem in the exterior of the Lax contour.

Degenerate solutions. Degenerate solutions describe evolution of singular contours like growth of slits or cuts in the plane. Since the Poisson bracket $\{z,\bar{z}\}$ vanishes, it might seem from (24) that the velocity of the contour vanishes as well and so there is no dynamics at all. In fact this is not exactly the case: V_n does vanish unless $z'(w) = 0$. We see that the growth is possible only at the points that are images of the critical points of the conformal map $z(w)$ lying on the boundary (on the unit circle). This means that only endpoints of arcs can move while other boundary points remain fixed. We see that the Lax contour or at least a finite part of it degenerates into an arc of a fixed curve Γ swept twice (back and forth), or into a collection of such arcs, and the evolution consists in moving the endpoints of the arc along the same curve Γ (see an example in Fig. 2).

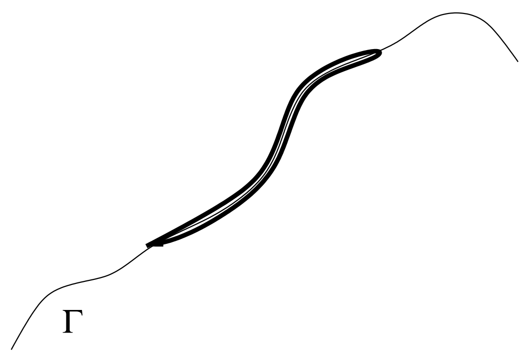

Figure 2: *A degenerate Lax contour. The arc can move along a fixed curve* Γ.

This agrees with the fact that for degenerate solutions the functions $z(w)$ and $\bar{z}(w^{-1})$ are algebraically dependent: the t_k-independent relation $R(z(w), \bar{z}(w^{-1})) = 0$ is just the equation of the curve Γ. The deformation parameters t_k do not admit so transparent interpretation as in the non-degenerate case.

5 Reductions of rank 1 and radial Löwner equation

Any particular solution of the dToda hierarchy can be regarded as a transition from the variables $t_0, t_1, t_2, \ldots$ to the variables $r, a_0, a_1, \ldots$ which are coefficients of the Lax function $z(w)$: $z(w) = rw + a_0 + a_1 z^{-1} + \ldots$. The non-degenerate solutions result in true changes of variables, i.e., the Jacobian of this transition does not vanish. For degenerate solutions, the Jacobi matrix is degenerate. In this section we present a detailed analysis of the simplest nontrivial case when this matrix is of rank 1. We say that the corresponding solutions are *reductions of rank 1* (of the dToda hierarchy). As is easy to see, the reduction of rank 1 implies that the Lax function and thus $w(z)$ depends on all the times t_j through *only one independent function* $q = q(\{t_j\})$:

$$z(w; \{t_j\}) = z(w, q)\,, \quad w(z; \{t_j\}) = w(z, q) \tag{33}$$

Our goal is to characterize the form of the function $w(z) = w(z, q)$ or $z(w) = z(w, q)$ consistent with the dToda hierarchy. Without loss of generality, we set $q := \log r$. We shall see that the consistency condition is the *radial Löwner equation* [7]

$$\frac{\partial w(z)}{\partial q} = w(z)\, \frac{\eta(q) + w(z)}{\eta(q) - w(z)} \tag{34}$$

where $\eta(q)$ is arbitrary continuous function of q such that $|\eta(q)| = 1$ (the "driving function"). This equation is well known in the theory of univalent functions (see, e.g., [21]) as a differential equation for conformal maps of slit domains parameterized by a parameter q.

In the calculation below, we closely follow [12]. Let us plug the ansatz (33) into the equation (13). Using the chain rule of differentiation, we have:

$$D(z_1)q \cdot \partial_q \log(e^q w(z_2)) = -\,\frac{\partial_q(e^q w(z_1)) - \partial_q(e^q w(z_2))}{e^q(w(z_1) - w(z_2))}\, \partial_{t_0} q$$

Now, using (11) and assuming that $\partial_{t_0} q \neq 0$, we get the relation

$$\frac{1}{2}\partial_q \log(e^q w(z_1))\, \partial_q \log(e^q w(z_2)) = \frac{w(z_1)\partial_q \log(e^q w(z_1)) - w(z_2)\partial_q \log(e^q w(z_2))}{w(z_1) - w(z_2)} \tag{35}$$

which means that the combination

$$\eta(q) = -w(z)\frac{1 + \partial_q \log w(z)}{1 - \partial_q \log w(z)} \tag{36}$$

does not depend on z. This implies the radial Löwner equation (34) or, for the Lax function $z(w, q)$,

$$\frac{\partial z(w)}{\partial q} = -\, w\, \frac{\eta(q) + w}{\eta(q) - w}\, \frac{\partial z(w)}{\partial w} \tag{37}$$

Note that the functions $1/\bar{w}(z)$ and $\bar{z}(w^{-1})$ obey the same Löwner equations (34) and (37) respectively.

Equation (14) implies that $|\eta(q)| = 1$. Indeed, with the reduction imposed it becomes

$$\frac{1}{2}\, \partial_q \log(e^q \bar{w}(\bar{z}_1))\, \partial_q \log(e^q w(z_2)) = \frac{\partial_q \log(\bar{w}(\bar{z}_1) w(z_2))}{1 - \bar{w}(\bar{z}_1) w(z_2)}$$

or, after rearranging,

$$\bar{w}(\bar{z}_1)\, \frac{1 + \partial_q \log \bar{w}(\bar{z}_1)}{1 - \partial_q \log \bar{w}(\bar{z}_1)}\, w(z_2)\, \frac{1 + \partial_q \log w(z_2)}{1 - \partial_q \log w(z_2)} = 1$$

that just means that $\overline{\eta(q)}\eta(q) = 1$. In fact this constraint follows already from eq. (36): let z belong to the image of the unit circle under the map $z(w)$ (i.e., to the Lax contour), then complex conjugation of (36) yields $\overline{\eta(q)} = \eta^{-1}(q)$.

Using the Löwner equations for $z(w)$ and $\bar{z}(w^{-1})$, it is straightforward to verify that $\{z(w), \bar{z}(w^{-1})\} = 0$. As it was argued in section 3, this means a t_k-independent relation $R(z, \bar{z}) = 0$ between the Lax functions z and $\bar{z}$ which defines a curve Γ in the plane. Let us represent it in the form $\bar{z} = S_\Gamma(z)$. The function S_Γ is called the Schwarz function of the curve Γ [22]. For the degenerate solution of rank 1 under consideration it is an integral of motion. It is easy to see that the canonical transformation (19) corresponding to this solution can be written in terms of the Schwarz function as follows:

$$\bar{z} = 1/S_\Gamma(z)\,, \quad \bar{M} = -\,\frac{S_\Gamma(z)}{z S'_\Gamma(z)}\, M \tag{38}$$

The reality constraint (20) follows from the identity $\bar{S}_\Gamma(S_\Gamma(z)) = z$ obeyed by the Schwarz function.

As w sweeps the unit circle, $z(w)$ sweeps an arc of the curve Γ (back and forth). The arc depends on all the times through q. The function $z(w)$ conformally maps the exterior of the unit circle onto the complement of the arc. We see that this map does obey the Löwner equation as it must.

The dependence of q on the times t_k, $\bar{t}_k$ is determined by a system of equations of hydrodynamic type. They follow from eq. (11) which can be written as $D(z)q = -\frac{1}{2}(1 + \partial_q \log w(z))\partial_{t_0} q$. Using the Löwner equation, we obtain:

$$D(z)q = \frac{\eta(q)}{w(z) - \eta(q)}\, \partial_{t_0} q \tag{39}$$

From (5) we conclude that

$$\frac{\eta}{w(z)-\eta} = \sum_{k\geq 1} \frac{z^{-k}}{k}\, \phi_k(\eta)$$

with $\phi_k(w)$ as in (27) and thus the system of equations of hydrodynamic type reads

$$\partial_{t_k} q = \phi_k(\eta(q))\partial_{t_0} q\,, \quad k = 1, 2, \ldots \tag{40}$$

Equations containing $\bar{t}_k$-derivatives are obtained by complex conjugation.

At last, it should be mentioned that the chordal version of the Löwner equation (see, e.g., [13]), emerges, in a similar way, in the context of the dispersionless KP hierarchy [8, 9, 10].

6 The large N Dyson gas representation of solutions to the dToda hierarchy

In this section, we reconstruct the solutions of the dToda hierarchy (both non-degenerate and degenerate) using the Dyson gas representation, i.e., eigenvalue versions of matrix integrals for certain models of random matrices.

Consider the following N-fold integral over the complex plane:

$$\tau_N = \frac{1}{N!}\int_{\mathbb{C}} \prod_{m<n} |z_m - z_n|^2 \prod_{j=1}^{N} e^{\frac{1}{\hbar}\sum_{k\geq 1}(t_k z_j^k + \bar{t}_k \bar{z}_j^k)}\, d\mu(z_j, \bar{z}_j) \tag{41}$$

where $d\mu$ is some integration measure and $\hbar$ is a parameter. For $d\mu = e^{-\frac{1}{\hbar}U(z,\bar{z})}d^2 z$ the integral is equal to the partition function of the model of normal random matrices with the potential $2\mathcal{R}e \sum_k t_k z^k - U(z,\bar{z})$ written as an integral over eigenvalues. Equivalently, it is equal to the partition function of the system of 2D Coulomb charges interacting via the logarithmic potential in an external field (the Dyson gas). It appears that both Laplacian growth and growth of slit domains can be simulated by the large N limit of this integral.

The basic fact linking the integral (41) to integrable systems is that for any measure $d\mu$ (including singular measures supported on sets of dimension less than 2), τ_N, as a function of $\{t_k\}$, $\{\bar{t}_k\}$, is a τ-function of the 2D Toda hierarchy with a nonzero dispersion parameter proportional to $\hbar$, i.e., it obeys the full set of Hirota bilinear identities for this hierarchy [23, 6]. In a slightly different form, this statement first appeared in [24], see also [25]. The dispersionless version is reproduced in the large N limit such that $N \to \infty$, $\hbar \to 0$, and $t_0 = \hbar N$ remains finite. Then τ_N generates the dispersionless "τ-function" (or rather "free energy") $\mathcal{F}$ via

$$\mathcal{F}(t_0, \{t_k\}, \{\bar{t}_k\}) = \lim_{N\to\infty}\left(\hbar^2 \log \tau_N\right) \tag{42}$$

It obeys the dispersionless Hirota relations (see below).

Second order t_k-derivatives of $\mathcal{F}$ enjoy a nice geometric interpretation through conformal maps. This goes as follows. As $N \to \infty$, the integral (41) is determined by the most

favorable configuration of z_i's, i.e., the one at which the integrand has a maximum. Using the electrostatic analogy, one can see that this holds when the points z_i (2D Coulomb charges) densely fill a bounded domain D in the complex plane. In terms of the mean density of the charges $\langle\rho(z)\rangle = \hbar\left\langle\sum_k \delta^{(2)}(z-z_k)\right\rangle$ this domain is characterized by the condition

$$\lim_{N\to\infty}\langle\rho(z)\rangle > 0 \quad \text{if } z\in\mathsf{D} \quad \text{and} \quad \lim_{N\to\infty}\langle\rho(z)\rangle = 0 \quad \text{otherwise}$$

For simplicity, we assume that D is connected. In the matrix model interpretation, this domain is called the support of eigenvalues.

Let $w(z)$ be the conformal mapping function from the exterior of the domain D onto the exterior of the unit circle normalized as $w(z) = z/r + O(1)$ at large $|z|$ with a real r called the exterior conformal radius of the domain D. In [26, 19] it was shown that the function $w(z)$ can be expressed through $\mathcal{F}$ in the following different but equivalent ways:

$$rw(z) = z\, e^{-D(z)\partial_{t_0}\mathcal{F}} \tag{43}$$

$$rw(z) = z - a - D(z)\partial_{t_1}\mathcal{F} \tag{44}$$

$$rw^{-1}(z) = D(z)\partial_{\bar{t}_1}\mathcal{F} \tag{45}$$

where

$$2\log r = \frac{\partial^2\mathcal{F}}{\partial t_0^2}\,, \quad a = \frac{\partial^2\mathcal{F}}{\partial t_0\partial t_1} \tag{46}$$

and the operator $D(z)$ is defined in section 2. The consistency of these relations follows from equations of the dToda hierarchy. The 2D dToda hierarchy can be written in a generating form as

$$D(z_1)D(z_2)\mathcal{F} = \log\frac{rw(z_1) - rw(z_2)}{z_1 - z_2} \tag{47}$$

$$-D(z_1)\bar{D}(z_2)\mathcal{F} = \log\left(1 - \frac{1}{w(z_1)\bar{w}(z_2)}\right) \tag{48}$$

together with complex conjugate equations (cf. eqs. (13), (14) which are t_0-derivatives of (47), (48)).

We emphasize that all the relations given above hold true for *any* measure $d\mu$ in (41) provided the most favorable configuration of z_i's at $N\to\infty$ is well defined. If the measure is smooth, say $d\mu = e^{-\frac{1}{\hbar}U(z,\bar{z})}d^2z$, then this construction gives non-degenerate solutions to the dToda hierarchy discussed in section 3. It is easy to see that $U(z,\bar{z})$ is just the generating function of the canonical transformation (21), hence the notation. Singular measures $d\mu$ lead to degenerate solutions. In particular, one may consider the measure supported on a curve Γ, then the integral (41) becomes one-dimensional (along Γ) in each variable:

$$\tau_N = \frac{1}{N!}\int_\Gamma \prod_{m<n}|z_m - z_n|^2 \prod_{j=1}^{N} e^{\frac{1}{\hbar}\sum_{k\geq 1}(t_k z_j^k + \bar{t}_k\bar{z}_j^k)}\,|dz_j| \tag{49}$$

In the large N limit, the support of eigenvalues, D, is then an arc of the curve Γ (or several disconnected arcs). The function $w(z)$ maps the slit domain $\mathbb{C}\setminus\mathsf{D}$ onto the exterior of

the unit circle. The choice of the measure supported on a curve means a reduction of the dToda hierarchy. A familiar example is the dToda chain, where one may take Γ to be either real or imaginary axis. Consider a general (continuous) curve Γ infinite in both directions. It is clear that $w(z)$ and the Lax functions (the functions inverse to $w(z)$ and $\bar{w}(z)$) depend on the times through two parameters only. One can set them to be, for example, the positions of the two ends of the arc D on the curve Γ. This is a reduction of rank 2. The simplest way to obtain a reduction of rank 1 is to take the measure $d\mu$ supported on a half-infinite curve starting at a point z_0 such that the arc D always starts at z_0 as the times independently vary in some open set.

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